

CHARACTERISTICS OF FINITE-Q
TRANSMISSION LINES

A THESIS

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CHARACTERISTICS OF FINITE-Q TRANSMISSION LINES

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TABLE OF SYMBOLS

SYMBOL	DEFINITION
α	Attenuation per mile
α'	Attenuation per mile at high frequencies
α_m	Attenuation per mile at the various frequencies at which the input impedance is a maximum or a minimum
a	Total attenuation (αS)
a'	Total attenuation ($\alpha' S$) at high frequencies
β	Phase shift per mile
b	Total phase shift (βS)
B	A constant, equal to $\sinh^2 a'$
C	Capacitance per mile of line
f	Any frequency
$(f_n)_{\min}$	Frequencies at which the input impedance of a short-circuited line is a minimum
$(f'_n)_{\min}$	Frequencies at which the input impedance of an open-circuited line is a minimum
$(f_1)_{\min}$	The lowest frequency at which the input impedance of a short-circuited line is a minimum
$(f_n)_{\max}$	Frequencies at which the input impedance of a short-circuited line is a maximum
$(f'_n)_{\max}$	Frequencies at which the input impedance of an open-circuited line is a maximum.
$(f_1)_{\max}$	The lowest frequency at which the input impedance of a short-circuited line is a maximum
G	Conductance per mile of line
j	$\sqrt{-1}$
K	A constant, equal to $2\pi S \sqrt{LC}$
L	Inductance per mile of line

SYMBOL	DEFINITION
M	A constant, equal to $\sinh 2a'$
n	An integer (1,2,3, ...)
θ_o	The angle of the characteristic impedance
$ \theta_o $	The absolute value of the angle of the characteristic impedance
θ_{so}	The angle of the input impedance of an open-circuited line
ϕ_{so}	$\tan^{-1} \frac{-\sin 2b}{\sinh 2a}$
$ \phi_{so} $	The absolute value of ϕ_{so}
θ_{ss}	The angle of the input impedance of a short-circuited line
ϕ_{ss}	$\tan^{-1} \frac{\sin 2b}{\sinh 2a}$
$ \phi_{ss} $	The absolute value of ϕ_{ss}
R	Resistance per mile of line
R_{ss}	The resistive component of the input impedance of a short-circuited line
R_{so}	The resistive component of the input impedance of an open-circuited line
S	Length of line in miles
ω	$2 \pi f$
X_{ss}	The reactive component of the input impedance of a short-circuited line
X_{so}	The reactive component of the input impedance of an open-circuited line
Z_o	The characteristic impedance of a line
$ Z_o $	The magnitude of the characteristic impedance

SYMBOL	DEFINITION
$ Z_o _n$	The magnitude of the characteristic impedance at the various frequencies at which the input impedance is a maximum or a minimum
Z_{ss}	The input impedance of a short-circuited line
$ Z_{ss} $	The magnitude of the input impedance of a short-circuited line
$(Z_{ss})_{\max}$	A maximum value of the input impedance of a short-circuited line
$ Z_{ssn} _{\max}$	Magnitudes of specific maxima of the input impedance of a short-circuited line
$(Z_{ss})_{\min}$	A minimum value of the input impedance of a short-circuited line
$ Z_{ssn} _{\min}$	Magnitudes of specific minima of the input impedance of a short-circuited line
Z_{so}	The input impedance of an open-circuited line
$ Z_{so} $	The magnitude of the input impedance of an open-circuited line
$(Z_{so})_{\max}$	A maximum value of the input impedance of an open-circuited line
$ Z_{son} _{\max}$	Magnitudes of specific maxima of the input impedance of an open-circuited line
$(Z_{so})_{\min}$	A minimum value of the input impedance of an open-circuited line
$ Z_{son} _{\min}$	Magnitudes of specific minima of the input impedance of an open-circuited line

CHARACTERISTICS OF FINITE-Q TRANSMISSION LINES

CHAPTER I

INTRODUCTION

The infinite-Q transmission line, or lossless line, has been treated extensively in the literature.^{1,2,3,4,5} The finite-Q transmission line, or the line which has losses, has been analyzed on the basis of a fixed frequency and a variable length of line.^{6,7,8,9} This thesis presents an analysis of the finite-Q line on the basis of a fixed length of line and a variable frequency. Only the cases of a short-circuited and an open-circuited termination are considered. This work

¹Ware, L.A. and H.R. Reed, Communications Circuits (New York: John Wiley & Sons, Inc., 1949), pp. 108-115 and pp. 205-242.

²Johnson, W.C., Transmission Lines and Networks (New York: McGraw-Hill Book Company, Inc., 1950), pp. 137-150 and pp. 152-167.

³Weinbach, M.P., Principles of Transmission in Telephony (The Macmillan Company, 1924), pp. 235-248.

⁴Karakash, J.J., Transmission Lines and Filter Networks (The Macmillan Company, 1950), pp. 37-144.

⁵Sarbacher, R.I. and W. A. Edson, Hyper and Ultrahigh Frequency Engineering, (New York, John Wiley & Sons, Inc., 1948), pp. 334-341.

⁶Weinbach, M.P., op. cit., pp. 47-49.

⁷Ware, op. cit., pp. 91-108.

⁸Johnson, op. cit., pp. 132-137.

⁹Karakash, op. cit., p. 106.

is primarily of importance to the field of communications.

Chapter II analyzes the variation of the magnitude of the input impedance of a short-circuited and an open-circuited transmission line. Particular attention is paid to the frequencies at which the maxima and minima of the input impedance occur. Chapter III and Chapter IV analyze the phase angle of the input impedance of the short-circuited and the open-circuited transmission line. Chapter III treats the low-loss line, and Chapter IV deals with the high-loss line. A useful distinction between low-loss lines and high-loss lines is developed in Chapter IV, where it is shown that a transition point based on the following condition is expedient for this analysis:

$$(\sinh 2a) (\tan |\theta_0|) \leq 1 \quad .$$

If a line has parameters such that this equation is satisfied at all frequencies, it is classified as a low-loss line; conversely, if a line has parameters such that this equation is not satisfied at all frequencies, it is classified as a high-loss line.

The distortionless line has characteristics that are quite different from those of the general finite-Q transmission line; in fact, in many respects it behaves like an infinite-Q line. For this reason, the distortionless line is analyzed in Chapter V.

CHAPTER II

VARIATION OF THE MAGNITUDES OF Z_{ss} AND Z_{so} WITH FREQUENCY

This chapter deals with the variation of the magnitude of the input impedance of a transmission line with frequency when the receiving end is (1) short circuited and (2) open circuited. In both cases it is well known that the input impedance passes through maximum and minimum values as the frequency is varied.¹⁰ In this chapter it is shown that, for infinite- Q lines, the frequencies which produce maxima (minima) of input impedance are integral multiples of the lowest frequency producing a maximum (minimum). However, for finite- Q lines, the frequencies which produce maxima (minima) of input impedance are not integral multiples of the lowest frequency which produces a maximum (minimum).

The Short-Circuited Line

The equation for the input impedance of a line with a short circuit located a distance S from the sending end is¹¹

$$Z_{ss} = Z_o \tanh (\alpha + j\beta)S \quad (1-1)$$

or

$$Z_{ss} = Z_o \frac{\sqrt{\sinh^2 a + \sin^2 b}}{\sqrt{\sinh^2 a + \cos^2 b}} \frac{\tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}}{\tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}} \quad (1-2)$$

¹⁰Johnson, op. cit., p. 157.

¹¹Ware, op. cit., p. 97

where Z_0 is the characteristic impedance of the line, $a = \alpha S$, and $b = \beta S$.

Inspection of, or differentiation of equation (1-2) shows that the maxima of Z_{ss} , $(Z_{ss})_{\max}$, occur for

$$b = (2n-1) \frac{\pi}{2}, \quad (1-3)$$

where $n = 1, 2, 3, \dots$. Also, the minima of Z_{ss} , $(Z_{ss})_{\min}$, occur for

$$b = n\pi. \quad (1-4)$$

In order to obtain equations for the frequencies at which maximum Z_{ss} and minimum Z_{ss} occur, b must be written as a function of frequency. This is accomplished by analysis of the propagation constant of a line which is given by the following equation:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (1-5)$$

Solving equation (1-5) for β yields¹²

$$\beta = \sqrt{\frac{1}{2} \left[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - (RG - \omega^2 LC) \right]} \quad (1-6)$$

Substituting equation (1-6) into equation (1-3) and solving the resulting equation for the frequency yield the following equation for the frequencies which produce maximum Z_{ss}

¹²Sarbacher, op. cit., p. 325.

$$(f_n)_{\max} = \frac{2n-1}{4S\sqrt{LC}} \sqrt{\frac{(2n-1)^2 + 4S^2RG/\pi^2}{(2n-1)^2 + S^2(RC+LG)^2/\pi^2LC}} \quad (1-7)$$

Then substitution of equation (1-6) into equation (1-4) yields for the frequencies which produce minimum Z_{ss}

$$(f_n)_{\min} = \frac{n}{2S\sqrt{LC}} \sqrt{\frac{n^2 + S^2RG/\pi^2}{n^2 + S^2(RC+LG)^2/4\pi^2LC}} \quad (1-8)$$

Equations (1-7) and (1-8) will now be examined for both infinite-Q and finite-Q lines.

The Infinite-Q Line. Since R and G are zero for lossless lines, equations (1-7) and (1-8) immediately reduce to

$$(f_n)_{\max} = \frac{2n-1}{4S\sqrt{LC}} \quad (1-9)$$

$$(f_n)_{\min} = \frac{n}{2S\sqrt{LC}}$$

$(f_1)_{\max}$ and $(f_1)_{\min}$ are the lowest frequencies which produce $(Z_{ss})_{\max}$ and $(Z_{ss})_{\min}$. Equations (1-9) show that $(f_n)_{\max}$ and $(f_n)_{\min}$ are integral multiples of $(f_1)_{\max}$ and $(f_1)_{\min}$, respectively.

When the conditions that R and G (and hence α) are zero are substituted into equation (1-2), this equation reduces to

$$Z_{ss} = jZ_0 \tan b \quad (1-10)$$

Equation (1-10) shows that the magnitudes of the maxima of Z_{ss} are all infinite and that the magnitudes of the minima of Z_{ss} are all zero.

The Finite-Q Line. The radicals in equations (1-7) and (1-8) are of the form $\frac{x^2 + c}{x^2 + d}$, where $x = (2n-1)^2$ in equation (1-7) and $x = n^2$ in equation (1-8). In general, for each value of x , the radical will have a different value. As a consequence, $(f_n)_{\max}$ and $(f_n)_{\min}$ are not integral multiples of $(f_1)_{\max}$ and $(f_1)_{\min}$, respectively, for lines with finite losses.

Equations (1-7) and (1-8) reduce to a simple form at high frequencies. As n is increased by increasing the frequency, the first term in the numerator and in the denominator of the radicals becomes larger, until at high frequencies

$$\begin{aligned} (2n-1)^2 &\gg 4S^2RG/\pi^2 \\ (2n-1)^2 &\gg S^2(RC+LG)^2/\pi^2LC \end{aligned} \quad (1-11)$$

in equation (1-7). Similarly,

$$\begin{aligned} n^2 &\gg S^2RG/\pi^2 \\ n^2 &\gg S^2(RC+LG)^2/4\pi^2LC \end{aligned} \quad (1-12)$$

in equation (1-8). In both equations the radical then reduces to unity. This means that, if equations (1-11) and (1-12) hold true, the frequencies at which $(Z_{ss})_{\max}$ occur are, for practical purposes, equally spaced. Also, the frequencies at which $(Z_{ss})_{\min}$ occur are, similarly, equally spaced.

Examination of equations (1-11) and (1-12) reveals that a longer line, or a line with higher losses, requires a correspondingly higher frequency for equally-spaced frequencies of maximum and minimum Z_{ss} .

For the lossless line, it was found that the magnitudes of the

various maxima of Z_{ss} are equal and the the magnitudes of the various minima of Z_{ss} are equal. However, these statements are not true of the finite-Q line, as is evident from an examination of equation (1-2). For the lossless line, α is zero, and Z_o is a constant resistance at all frequencies. For a line with losses, however, both α and Z_o vary with frequency. At the frequencies at which Z_{ss} is a maximum, $\sin^2 b = 1$ and $\cos^2 b = 0$; so that the magnitude of $(Z_{ssn})_{\max}$ becomes

$$|Z_{ssn}|_{\max} = |Z_o|_n \cotanh \alpha_n S, \quad (1-13)$$

where $|Z_{ssn}|_{\max}$, $|Z_o|_n$, and α_n are the magnitudes of Z_{ss} , Z_o and α at the frequencies corresponding to n in equation (1-7).

The derivative of $|Z_o|$ with respect to ω is negative if $RC > LG$, which is the case for all lines of interest, with the exception of the distortionless line, for which $RC = LG$. The attenuation per mile is larger for higher frequencies, and thus $\cotanh \alpha_n S$ decreases with increasing frequency. Both factors in equation (1-13), therefore, tend to reduce the magnitude of $(Z_{ss})_{\max}$ at successively higher frequencies. It is to be noted that longer lengths (S) of the line produce a smaller magnitude of $(Z_{ssn})_{\max}$ for corresponding values of n .

At frequencies at which Z_{ss} is a minimum, $\sin^2 b = 0$ and $\cos^2 b = 1$; so that the magnitude of $(Z_{ssn})_{\min}$ becomes

$$|Z_{ssn}|_{\min} = |Z_o|_n \tanh \alpha_n S, \quad (1-14)$$

where $|Z_{ssn}|_{\min}$, $|Z_o|_n$, and α_n are the magnitudes of $(Z_{ss})_{\min}$, Z_o , and α at the frequencies corresponding to the various values of n in equation (1-8).

Since $|Z_0|$ decreases with increasing frequency while $\tanh \alpha_n S$ increases with increasing frequency, it is difficult to determine how $|Z_{ssn}|_{\min}$ varies with frequency. Figure 1, which is a plot of the variation of $|Z_{ss}|$ with frequency for an artificial line, indicates that these two opposite variations with frequency effectively nullify each other for this particular line, because there is very little variation in magnitude of the various minima of Z_{ss} . Figure 1 shows, however, that longer lengths (S) of the line produce larger values of $|Z_{ssn}|_{\min}$. This fact is also evident from equation (1-14) since the length (S) of the line affects only the factor $\tanh \alpha_n S$.

The Open-Circuited Line

Since the process for deriving equations for the variation of Z_{so} with frequency is the same as that discussed for the short-circuited line, the pertinent equations can be developed rapidly with a minimum of discussion.

The equation for the input impedance of a line with an open circuit at a distance S from the sending end is¹³

$$Z_{so} = Z_0 \cotanh (\alpha + j\beta)S \quad (1-15)$$

¹³Ware, op. cit., p. 96

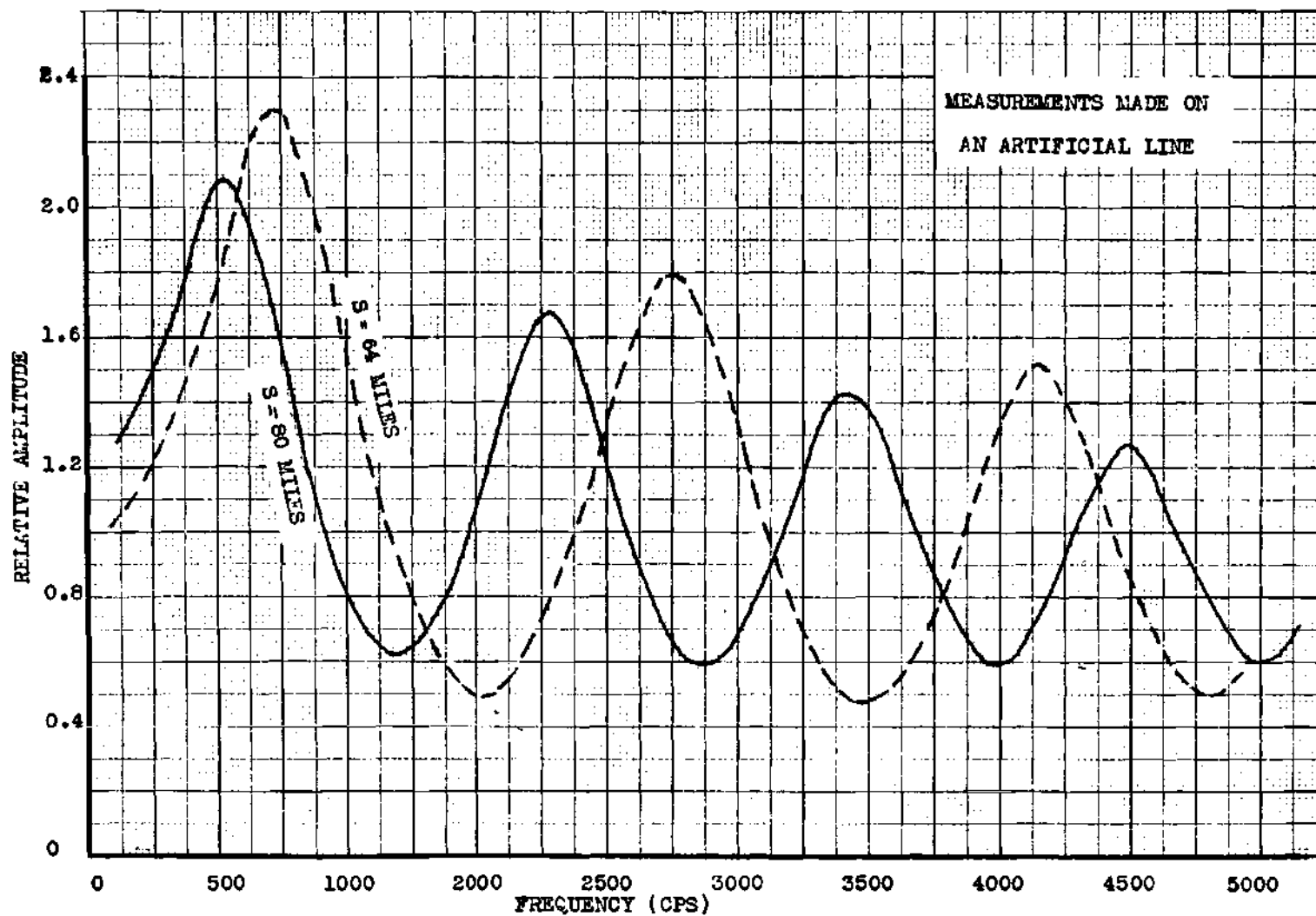


Fig. 1 VARIATION OF THE MAGNITUDE OF Z_{ss} WITH FREQUENCY

or

$$Z_{so} = Z_o \frac{\sqrt{\sinh^2 a + \cos^2 b}}{\sqrt{\sinh^2 a + \sin^2 b}} \frac{\tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}}{\tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}} \quad (1-16)$$

Inspection of, or differentiation of equation (1-16) shows that the maxima of Z_{so} occur for

$$b = n\pi \quad (1-17)$$

and the minima of Z_{so} occur for

$$b = (2n-1) \frac{\pi}{2} \quad (1-18)$$

The frequencies which produce maximum Z_{so} are then

$$(f'_n)_{\max} = \frac{n}{2S\sqrt{LC}} \sqrt{\frac{n^2 + S^2 RG/\pi^2}{n^2 + S^2(RC+LG)^2/4\pi^2 LC}}, \quad (1-19)$$

and the frequencies which produce minimum Z_{so} are

$$(f'_n)_{\min} = \frac{2n-1}{4S\sqrt{LC}} \sqrt{\frac{(2n-1)^2 + 4S^2 RG/\pi^2}{(2n-1)^2 + S^2(RC+LG)^2/4\pi^2 LC}}, \quad (1-20)$$

It is noted that maxima of Z_{so} occur at the frequencies which produce minima of Z_{ss} and that minima of Z_{so} occur at frequencies which produce maxima of Z_{ss} .

For a lossless line, equation (1-15) reduces to

$$Z_{so} = -j Z_o \cot b . \quad (1-21)$$

The frequencies which produce maximum and minimum Z_{so} are equally spaced. Furthermore, the magnitudes of the maxima of Z_{so} are all equal to infinity and the magnitudes of the minima of Z_{so} are all equal to zero.

In the case of the line with losses, the magnitudes of the maxima and minima of Z_{so} are found to vary with frequency in the same manner as the maxima and minima of Z_{ss} . Equation (1-16) yields the following expression for the magnitude of the maxima of Z_{so} :

$$|Z_{son}|_{\max} = |Z_o|_n \coth \alpha_n S , \quad (1-22)$$

where $|Z_{son}|_{\max}$, $|Z_o|_n$, and α_n are the magnitudes of Z_{so} , Z_o and α at the frequencies corresponding to the various values of n in equation (1-19).

Equation (1-16) yields the following equation for the minimum values of the magnitude of Z_{so} :

$$|Z_{son}|_{\min} = |Z_o|_n \tanh \alpha_n S , \quad (1-23)$$

where $|Z_{son}|_{\min}$, $|Z_o|_n$, and α_n are the magnitudes of Z_{so} , Z_o , and α at the frequencies corresponding to the various values of n in equation (1-20).

Since equations (1-22) and (1-23) show that $|Z_{son}|_{\max}$ and $|Z_{son}|_{\min}$ vary with frequency in the same manner as $|Z_{ssn}|_{\max}$ and $|Z_{ssn}|_{\min}$, it is evident that the magnitude of the maxima of Z_{so} decreases at higher values of frequency and that there is a tendency for the minima of Z_{so} to remain virtually constant.

A plot of the variation of the magnitude of Z_{so} with frequency for a finite-Q line would be similar to Figure 1. The maxima of Z_{so} , however, would occur where the minima of Z_{ss} occur, and the minima of Z_{so} would occur where the maxima of Z_{ss} occur.

CHAPTER III

PHASE ANGLE OF Z_{ss} AND Z_{so} FOR LOW-LOSS LINES

In this chapter it is shown that the frequencies at which Z_{ss} and Z_{so} are maximum and minimum do not coincide with the frequencies at which Z_{ss} and Z_{so} are purely resistive. Only lines with low losses are considered here; Chapter IV treats the case of the high-loss line.

The Short-Circuited Line

If equation (1-2) is written in slightly different form:

$$Z_{ss} = |Z_o| \frac{\sqrt{\sinh^2 a + \sin^2 b}}{\sqrt{\sinh^2 a + \cos^2 b}} \angle \theta_o + \tan^{-1} \frac{\sin 2b}{\sinh 2a} \quad (2-1)$$

it becomes evident that, for a line with losses, the frequencies at which Z_{ss} is a maximum or a minimum do not coincide with the frequencies at which θ_{ss} , the phase angle of Z_{ss} , vanishes. θ_{ss} vanishes when

$$\theta_{ss} = \theta_o + \tan^{-1} \frac{\sin 2b}{\sinh 2a} = 0. \quad (2-2)^{14}$$

Now the value of b which produces either a maximum Z_{ss} or a minimum Z_{ss} [see equations (1-3) and (1-4)] is always of such value as to make

$$\tan^{-1} \frac{\sin 2b}{\sinh 2a}$$

¹⁴Weinbach, op. cit., p. 86.

vanish. θ_{ss} , however, is not zero at these frequencies because θ is not zero. Figure 2 shows the variation of θ_0 with frequency for a typical finite-Q line.

It is noteworthy that, for infinite-Q lines, the frequencies which produce a maximum or a minimum of Z_{ss} also produce a purely resistive Z_{ss} , that is, the maxima and minima of Z_{ss} are pure resistances. This fact is evident if equation (2-2) is rearranged to yield

$$b = \frac{1}{2} \sin^{-1} (\tan |\theta_0| \sinh 2\alpha S) . \quad (2-3)^{15}$$

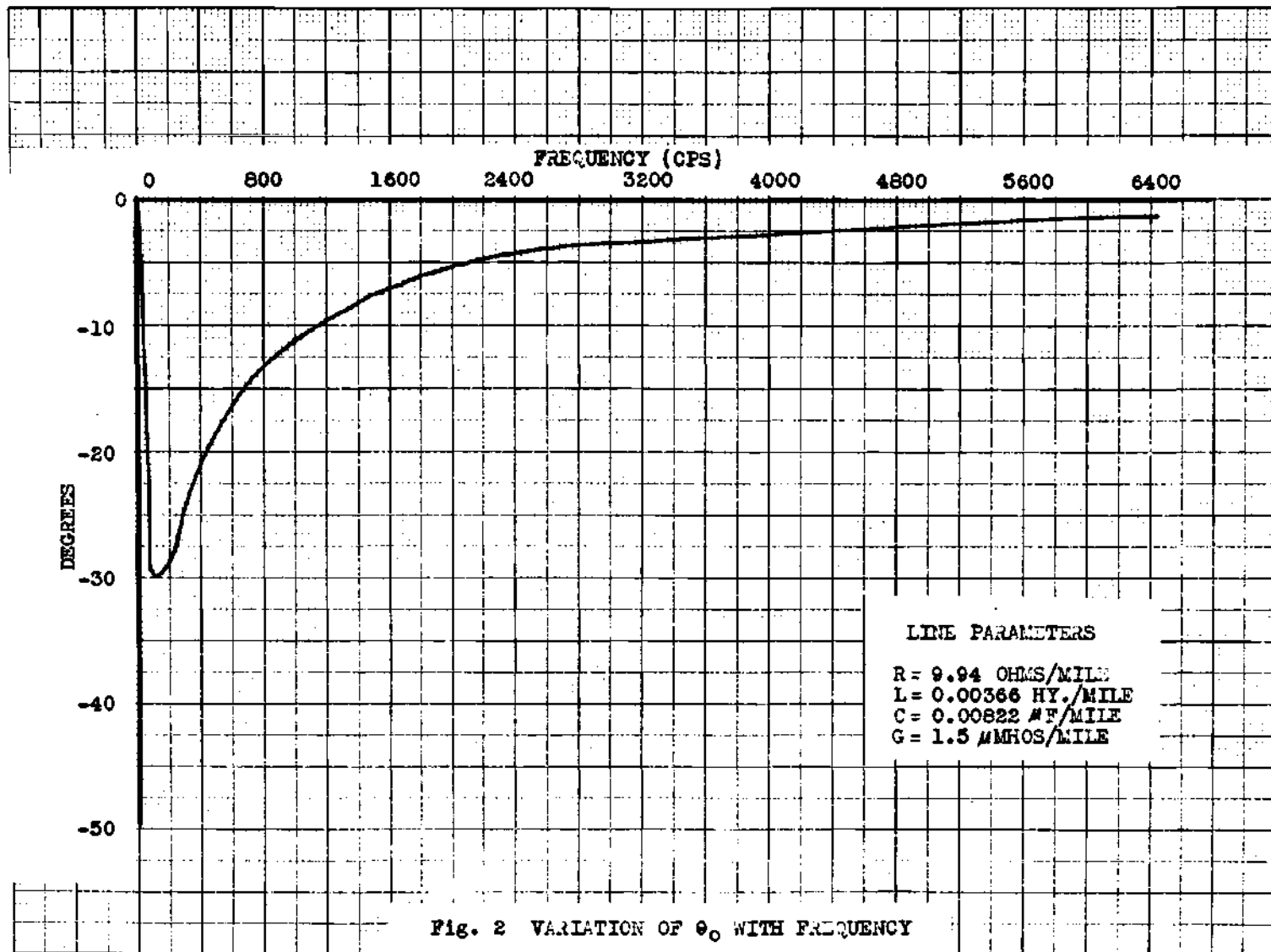
If $\alpha = 0$, this equation reduces to

$$b = \frac{1}{2} \sin^{-1} 0 = \frac{1}{2} n\pi . \quad (2-4)$$

Comparison of equation (2-4) with equations (1-3) and (1-4) reveals the fact that θ_{ss} becomes zero in the case of the lossless line at those frequencies for which Z_{ss} is a maximum and a minimum.

A clearer picture of the relationship between the frequencies at which the maxima and minima of Z_{ss} occur and the frequencies at which Z_{ss} is purely resistive for a finite-Q line is obtained if equation (1-1) is put in complex form. This is accomplished by expressing $\tanh (a + jb)$ as $\sinh (a + jb) / \cosh (a + jb)$. Upon using the trigonometric identities for the hyperbolic sine and the hyperbolic cosine of the sum of two angles and rationalizing the results, the following

¹⁵The use of $|\theta_0|$ in equation (2-3) follows from the fact that θ_0 is a negative angle.



equation results:

$$Z_{ss} = Z_0 \left[\frac{\sinh 2a}{2(\sinh^2 a + \cos^2 b)} + j \frac{\sin 2b}{2(\sinh^2 a + \cos^2 b)} \right]. \quad (2-5)$$

Equation (2-5) will be examined first at high frequencies. This leads to results which clarify the picture at low frequencies.

At high frequencies, α , β , and Z_0 approach the following constant values:¹⁶

$$\left. \begin{aligned} \alpha' &\cong \frac{R}{2} \sqrt{\frac{C}{L}} + \frac{G}{2} \sqrt{\frac{L}{C}} \\ \beta &\cong \omega \sqrt{LC} \\ Z_0 &\cong \sqrt{LC} \end{aligned} \right\} \quad (2-6)$$

When equations (2-6) are substituted into equation (2-5), this equation becomes

$$Z_{ss} = \frac{M \sqrt{\frac{L}{C}}}{2(B + \cos^2 Kf)} + j \frac{\sqrt{\frac{L}{C}} \sin 2Kf}{2(B + \cos^2 Kf)}, \quad (2-7)$$

where M , B , and K are constants given by

$$M = \sinh 2a'$$

$$B = \sinh^2 a'$$

$$K = 2\pi S \sqrt{LC}$$

The full significance of equation (2-7) in its present form is

¹⁶Johnson, op. cit., p. 47

not apparent. The two terms of this equation are parametric equations with the resistive and reactive components of Z_{ss} expressed as functions of frequency. Elimination of the frequency parameter yields

$$(R_{ss} - \sqrt{\frac{L}{C}} \coth 2a')^2 + X_{ss}^2 = \frac{L}{C} \operatorname{csch}^2 2a' \quad (2-8)$$

which is the equation of a circle with center at $R_{ss} = \sqrt{\frac{L}{C}} \coth 2a'$ and with radius equal to $\sqrt{\frac{L}{C}} \operatorname{csch} 2a'$. For large values of a' (large α' or large S) the radius of the circle approaches zero and the center of the circle approaches the value $R_{ss} = \sqrt{\frac{L}{C}}$.

As shown in equation (2-6), α , at high frequencies, approaches a constant value determined by the line constants, being larger for larger values of R and G . Figure 3 shows the circles represented by equation (2-8) for lines with the same values of L and C but with different values of R and G . (The same curves could be obtained by using a single line of constants R , L , G , and C and plotting Z_{ss} for various lengths of line, since the total attenuation is determined by both α and S .) This figure enables one to visualize the variation of Z_{ss} as the frequency is varied. At the lower frequencies where α is small, the locus of Z_{ss} will fall on one of the larger circles. As α increases with increasing frequency, this locus will then spiral around the point $R_{ss} = \sqrt{\frac{L}{C}} \coth 2a'$ and at high frequencies will approach the circle given by equation (2-8).

Figure 4 is a polar plot of Z_{ss} as a function of frequency for a typical telephone line. Z_{ss} is resistive at zero frequency. Because

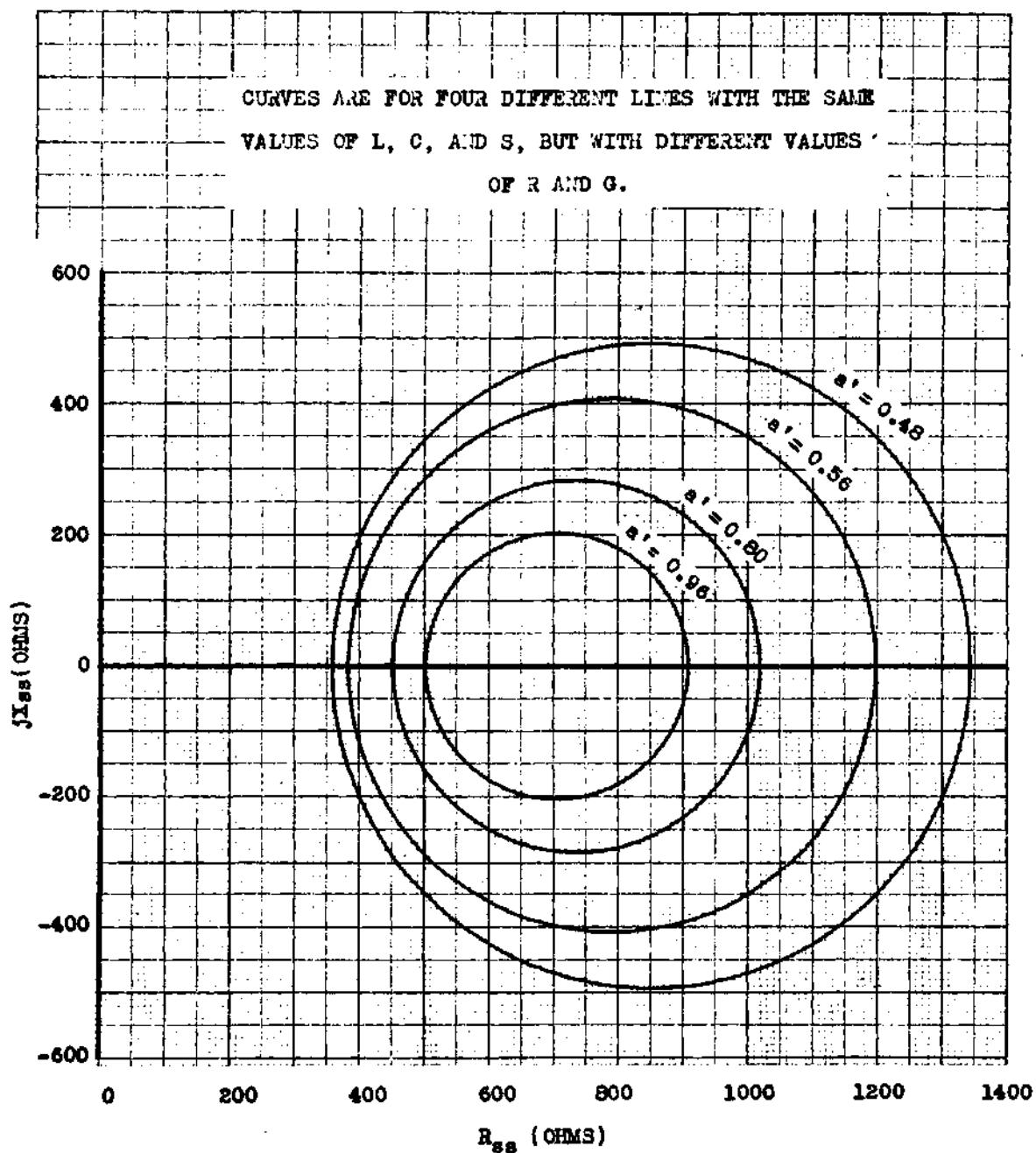


Fig. 3 VARIATION OF Z_{ss} AT HIGH FREQUENCIES

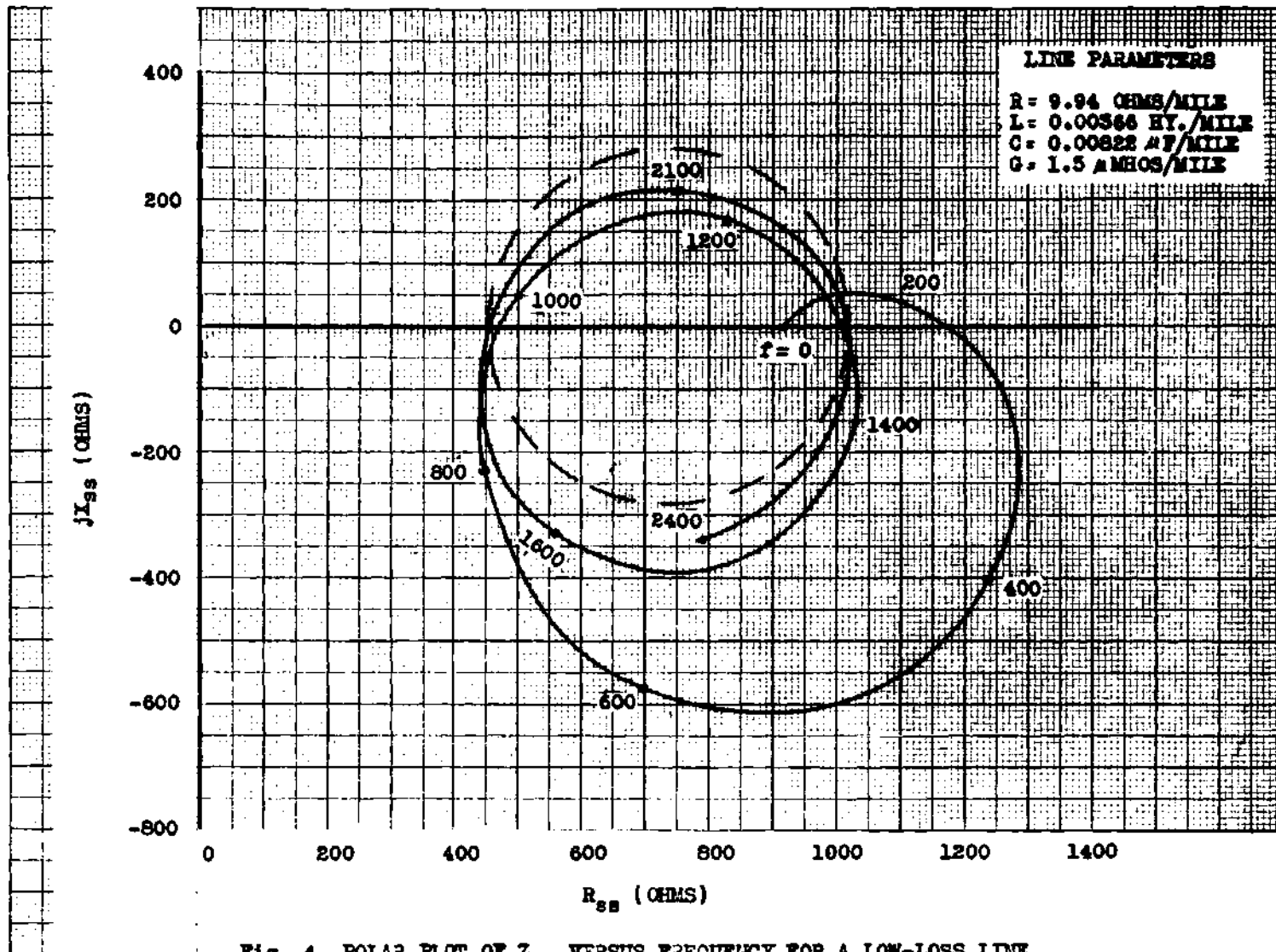


Fig. 4 POLAR PLOT OF Z_{sB} VERSUS FREQUENCY FOR A LOW-LOSS LINE

a shorted line is first inductive¹⁷ as the frequency is increased from zero, the first portion of the spiral is in the (+ j) region. The curve then spirals around as deduced from Figure 3.

Figure 4 shows again that, at the low frequencies, the frequency which produces maximum or minimum Z_{ss} does not make θ_{ss} equal to zero. It is apparent that, at the high frequencies, the frequencies which produce maximum and minimum Z_{ss} are negligibly different from those which produce a purely resistive Z_{ss} .

The Open-Circuited Line

Again, the treatment of the open-circuited line is similar to that for the short-circuited line; and, as in Chapter I, the equations can be developed with a minimum of discussion.

Equation (1-13) may be written in the following form:

$$Z_{so} = |Z_o| \frac{\sqrt{\sinh^2 a + \cos^2 b}}{\sqrt{\sinh^2 a + \sin^2 b}} \angle \theta_o + \tan^{-1} \left(\frac{-\sinh 2b}{\sinh 2a} \right) \quad (2-9)$$

Equation (2-9) shows that θ_{so} is zero when

$$\theta_{so} = \theta_o + \tan^{-1} \left(\frac{-\sinh 2b}{\sinh 2a} \right) = 0 \quad (2-10)$$

Equation (2-10) shows that for a finite-Q line, the maxima and minima of

¹⁷Shea, T. E., Transmission Networks and Wave Filters (New York: D. Van Nostrand, 1948), p. 148.

Z_{so} are not purely resistive because θ is not zero.

In order to examine the equation for Z_{so} at high frequencies, equation (1-15) is expanded by using the trigonometric identities for the hyperbolic sine and the hyperbolic cosine of the sum of two angles. Rationalizing the result yields

$$Z_{so} = Z_o \left[\frac{\sinh 2a}{2(\sinh^2 a + \sin^2 b)} - j \frac{\sin 2b}{2(\sinh^2 a + \sin^2 b)} \right]. \quad (2-11)$$

Applying equation (2-6) to equation (2-11) yields

$$Z_{so} = \frac{M\sqrt{\frac{L}{C}}}{2(B + \sin^2 Kf)} - j \frac{\sqrt{\frac{L}{C}} \sin 2Kf}{2(B + \sin^2 Kf)} \quad (2-12)$$

where M , B , and K are constants given by

$$M = \sinh 2a'$$

$$B = \sinh^2 a'$$

$$K = 2\pi S \sqrt{LC}.$$

Elimination of the frequency variable yields

$$(R_{so} - \sqrt{\frac{L}{C}} \coth 2a')^2 + X_{so}^2 = \frac{L}{C} \operatorname{csch}^2 2a', \quad (2-8)$$

which is the same equation of a circle obtained for the short-circuited line at high frequencies.

Figure 3, then, can be used to visualize the polar plot of Z_{so} as well as that of Z_{ss} . Since the impedance of an open-circuited line is first capacitive as the frequency is increased from zero, Z_{so} will begin in the $(-jX)$ region, spiral around the point $R_{so} = \sqrt{\frac{L}{C}} \coth 2a'$ as the

frequency increases, and at high frequencies approach the circle given by equation (2-8). A polar plot of Z_{s0} for a typical telephone line is given in Figure 5.

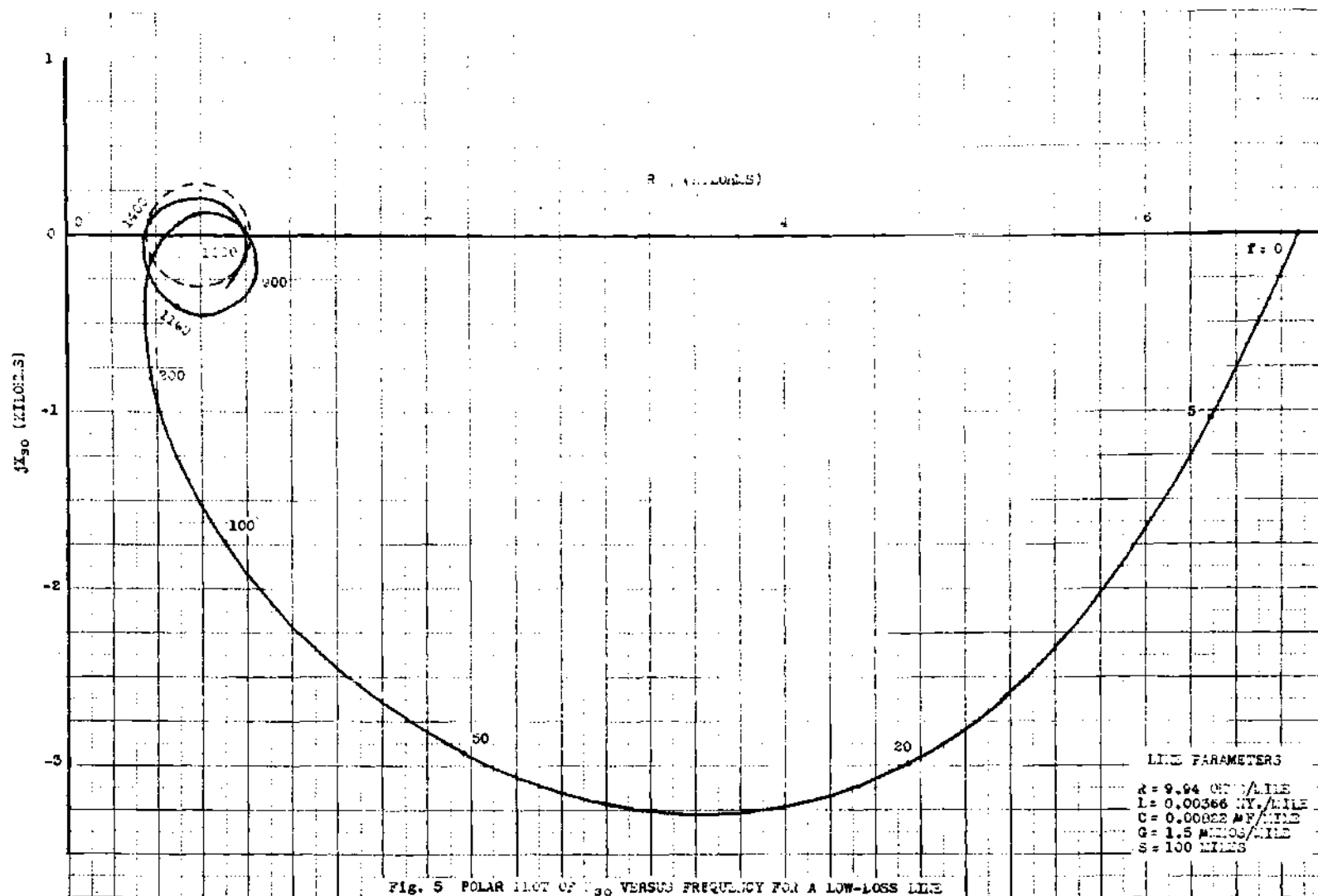


Fig. 5 POLAR PLOT OF Z_0 VERSUS FREQUENCY FOR A LOW-LOSS LINE

CHAPTER IV

PHASE RELATIONS FOR HIGH-LOSS LINES

Chapter III considered phase relations in low-loss line. This chapter extends the work to cover high-loss lines. It should be remembered that the total attenuation of a line is the product of the length of the line and the attenuation per unit length. From this viewpoint, a long line with small α may have high losses, and a short line with large α may have low losses.

Since the work in this chapter is based to a large extent upon the variation of the angle θ_0 with frequency and the variation of the attenuation per mile with frequency, these relations will be discussed now.

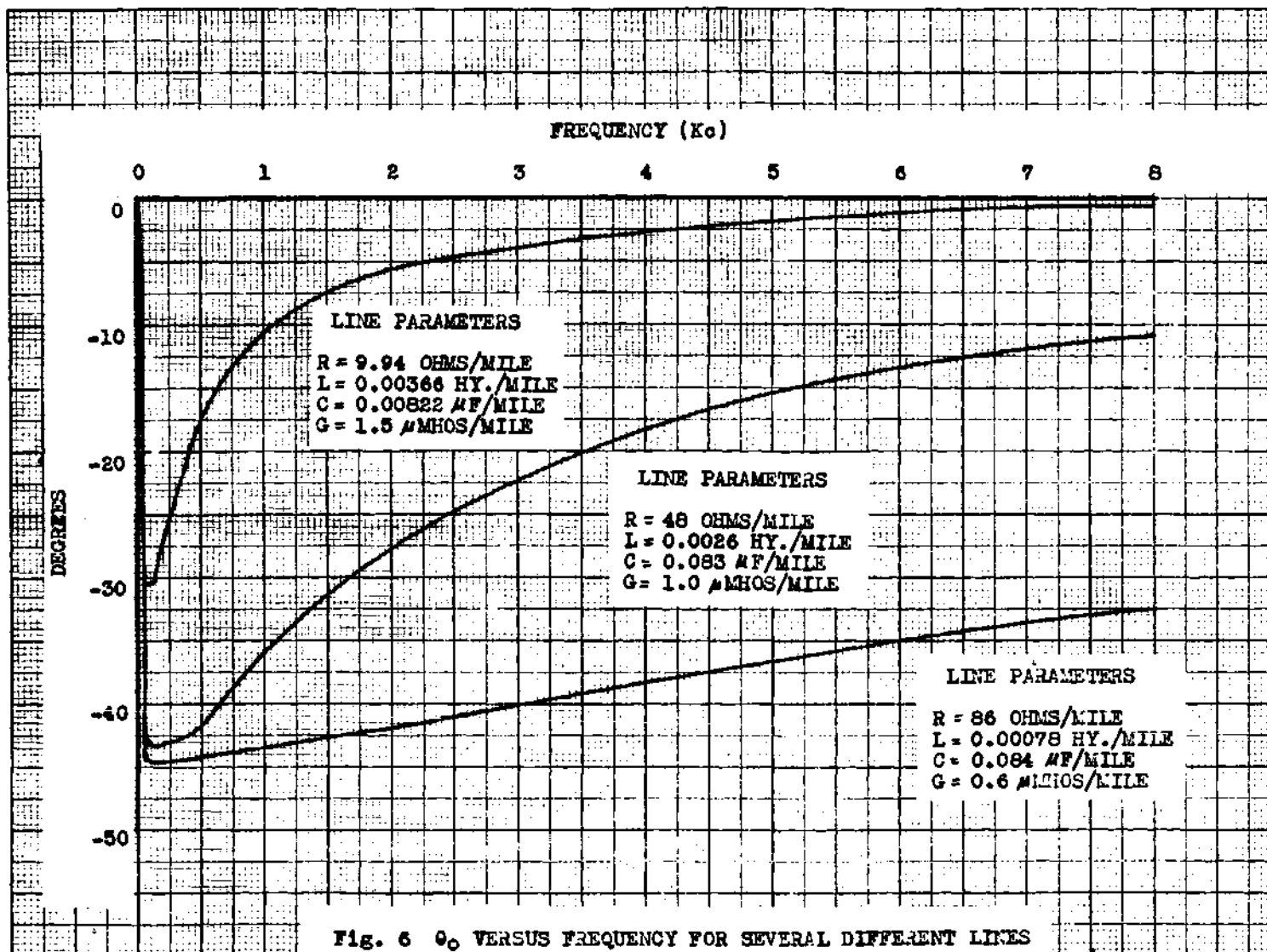
The Variation of θ_0 with Frequency

The equation for the angle θ_0 of the characteristic impedance Z_0 is

$$\theta_0 = \frac{1}{2} \tan^{-1} \frac{\omega(LG - RC)}{RG + \omega^2 LC} \quad (3-1)^{18}$$

In Figure 6 θ_0 is plotted as a function of frequency for several lines with different values of R , L , G , and C . θ is zero at zero frequency, reaches its most negative value at

¹⁸The condition $LG = RC$ will make θ vanish at all frequencies. This is the condition for a distortionless line. This line is discussed in Chapter IV.



$$f = \frac{1}{2\pi \sqrt{\frac{LC}{RG}}} , \quad (3-2)$$

and approaches zero at high frequencies. θ_o approaches zero more slowly for lines with larger values of R .

The Variation of α with Frequency¹⁹

From equation (1-5), the equation for the attenuation of a line per mile is

$$\alpha = \sqrt{\frac{1}{2} \left[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + (RG - \omega^2 LC) \right]} . \quad (3-3)$$

The plot of α as a function of frequency is given in Figure 7 for several lines with various values of R , L , G , and C . At the high frequencies, α approaches a constant value, the value approached being larger for lines with larger values of R .

The Short-Circuited Line

As shown in Chapter III, the input impedance of a short-circuited line is purely resistive when

$$\theta_{ss} = \theta_o + \tan^{-1} \frac{\sin 2b}{\sinh 2a} = 0 . \quad (2-2)$$

¹⁹Johnson, op. cit., p. 48

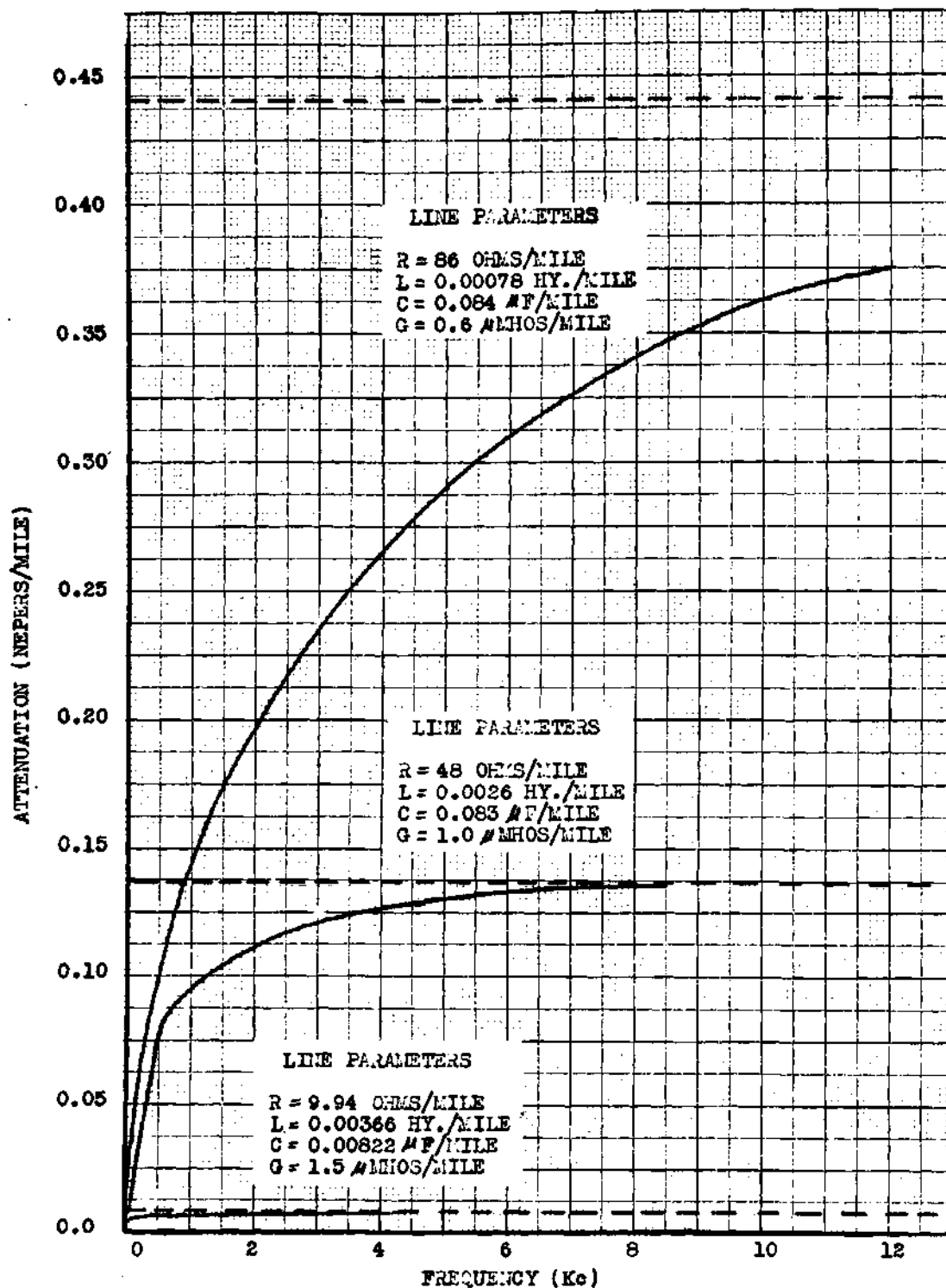


Fig. 7 α VERSUS FREQUENCY FOR SEVERAL DIFFERENT LINES

Taking the tangent of both angles yields

$$\sin 2b = -\tan \theta_0 \sinh 2a . \quad (3-4)$$

Since θ_0 is negative for all physical lines of interest, equation (3-4) may be written in the following form

$$\sin 2b = \tan |\theta_0| \sinh 2a . \quad (3-5)$$

Because $\sin 2b$ never exceeds unity, equation (3-5) states that

$$\tan |\theta_0| \sinh 2a \leq 1 . \quad (3-6)$$

There are lines, however, with parameters such that condition (3-6) is not satisfied at all frequencies. As a matter of fact, for any physically-realizable line it is possible to take a length sufficient to cause the product of $\tan |\theta_0|$ and $\sinh 2a$ to exceed unity. When this product is greater than unity, it is clear from the foregoing analysis that

$$\tan^{-1} \frac{\sin 2b}{\sinh 2a}$$

cannot be equal in magnitude and opposite in sign to θ_0 , which is the requirement stated in equation (2-2) for θ_{ss} to be zero.

This phenomenon can be shown conveniently on a plot if equation (3-6) is written in the following form:

$$\sinh 2a \leq \cotan |\theta_0| . \quad (3-7)$$

Figure 8 is a plot of $\cotan |\theta_0|$ and $\sinh 2a$ as functions of frequency

for a typical telephone line; $\sinh 2a$ is plotted for various lengths of line to illustrate the effects of total attenuation.

The intersections of the curves of $\cotan |\theta_o|$ and $\sinh 2a$ mark the frequency ranges over which condition (3-6) is satisfied. For lengths of this particular line which are less than approximately 110 miles, condition (3-6) is satisfied at all frequencies. For lengths of this line greater than approximately 110 miles, however, equation (3-6) is satisfied only at very low and at high frequencies, there being a frequency range in between for which $\sinh 2a$ is greater than $\cotan |\theta_o|$.

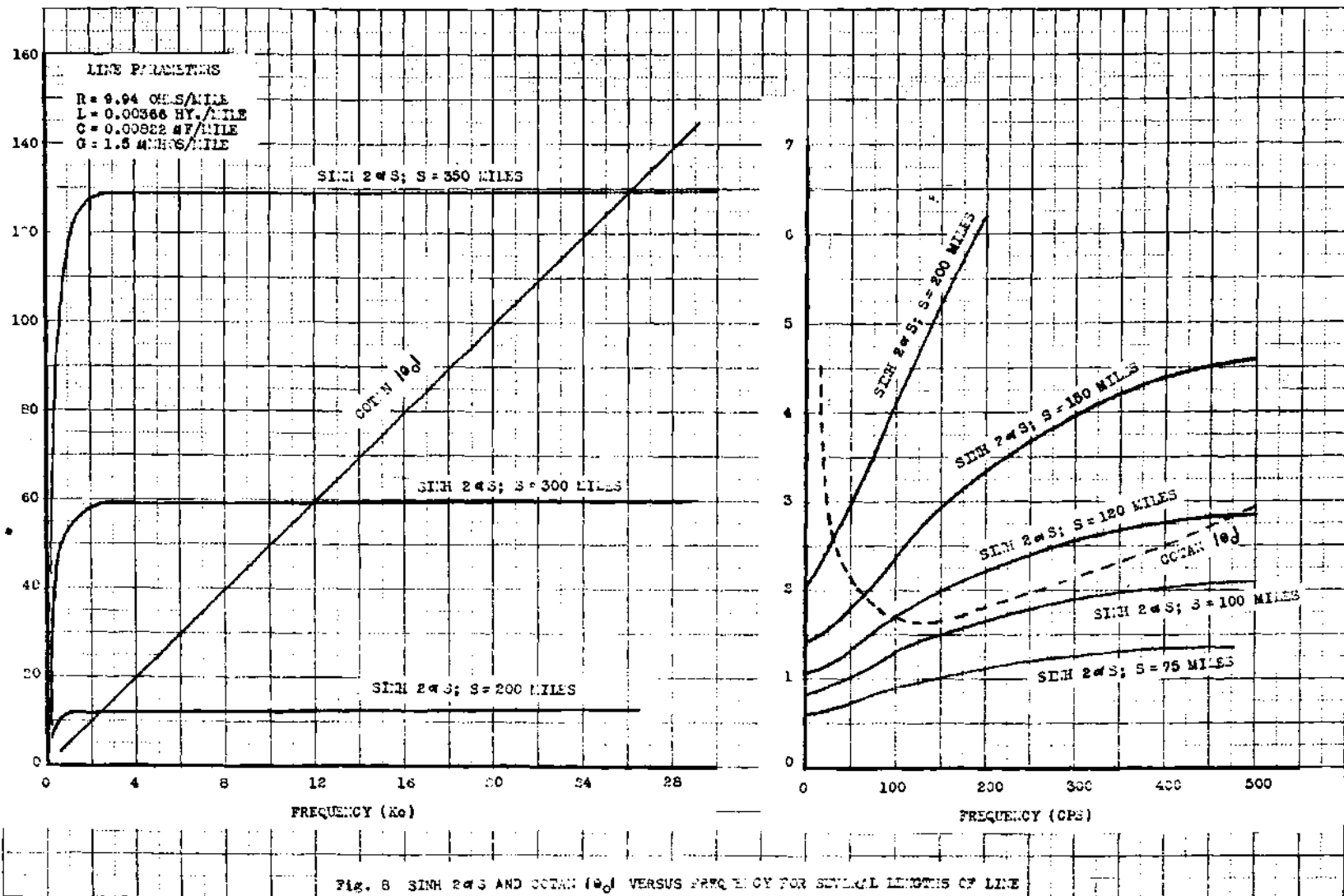
It can now be shown that θ_{ss} does not vanish in the low-frequency range for which equation (3-6) is satisfied. Figure 8 shows that, for lengths of this particular line greater than 120 miles, the highest frequency in this low-frequency region for which θ_{ss} can vanish is the frequency of maximum θ_o ,

$$f = \frac{1}{2\pi \sqrt{\frac{LC}{RC}}} \quad (3-2)$$

Figure 9 shows the variation of $(-\theta_o)$ and

$$\phi_{ss} = \tan^{-1} \frac{\sin 2b}{\sinh 2a}$$

over this low-frequency region. Since the curves of $(-\theta_o)$ and ϕ_{ss} do not intersect, there is no frequency in this region at which θ_{ss} becomes zero. It is now apparent that, for a high-loss line, frequencies at which θ_{ss} becomes zero occur only in the high frequency range for which equation (3-6) is satisfied.



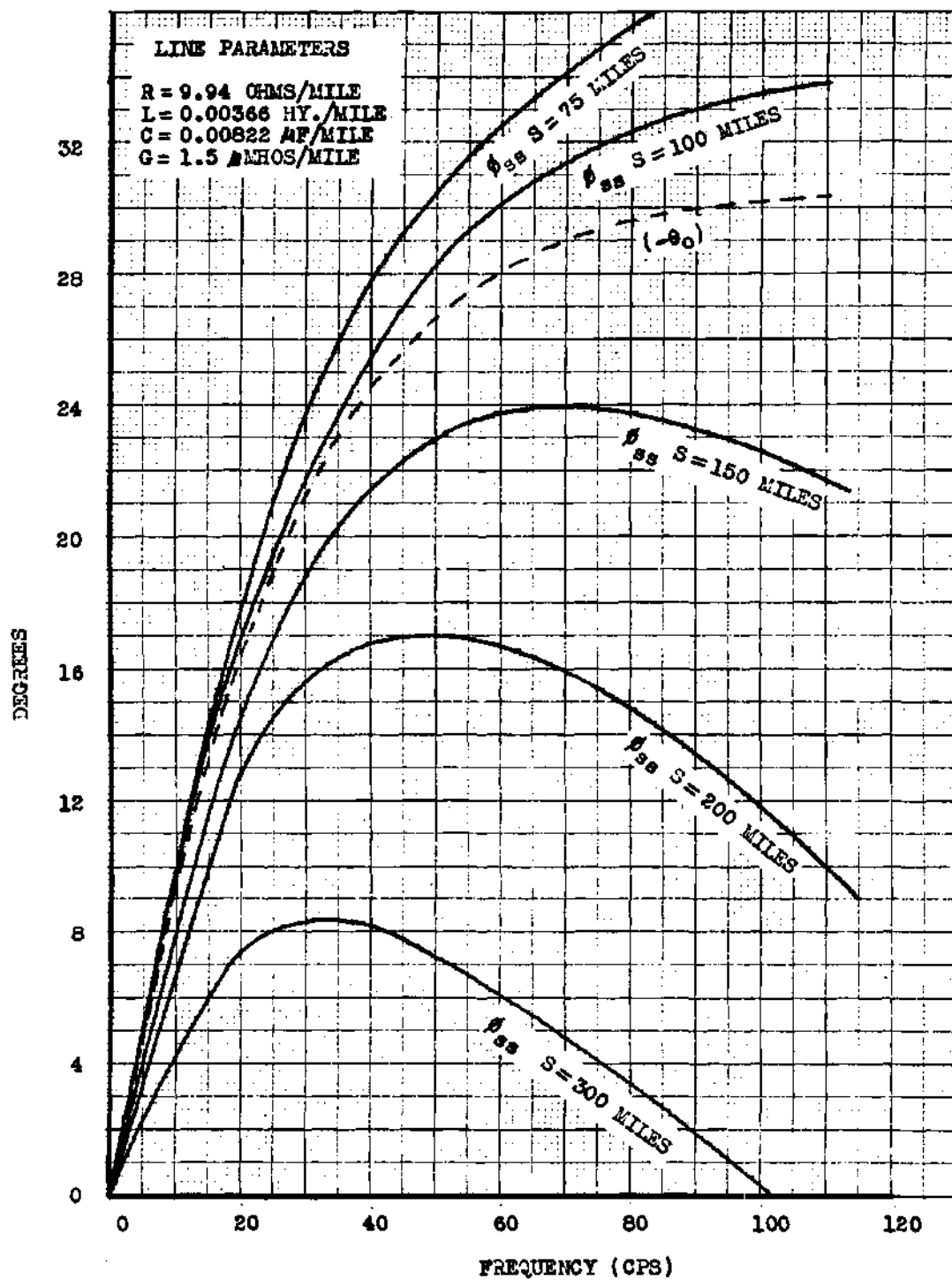


Fig. 9 VARIATION OF $(-\theta_0)$ AND ϕ_{ss} AT THE LOWEST FREQUENCIES

A singularly clear picture of the frequencies at which θ_{ss} vanishes is obtained by extending the abscissa of Figure 9. Thus, in Figure 10 $(-\theta_o)$ and ϕ_{ss} are plotted over a wider frequency range. ϕ_{ss} is plotted for two different lengths of the line, one length ($S = 100$ miles) which produces a low-loss line and one length ($S = 200$ miles) which produces a high-loss line. The intersections of the curves of $(-\theta_o)$ and ϕ_{ss} indicate the frequencies at which θ_{ss} is zero.

It is noted that the effect of high total attenuation on ϕ_{ss} is to decrease its range of variation. This explains why it is impossible, with a high-loss line for θ_{ss} to vanish at the low frequencies.

The polar plot of the variation of the input impedance of a short-circuited, high-loss telephone line is shown in Figure 11. The magnitude varies through various maxima and minima at the low frequencies even though the phase angle of Z_{ss} does not pass through zero. For clarity, only three complete loops are shown. The curve continues to loop as the frequency is increased and at high frequencies approaches the dotted circle which is given by equation (2-8).

Figure 11 shows that the input impedance is capacitive at the lowest frequencies. It was stated in Chapter III that, in general, the input impedance of a short-circuited transmission line is first inductive as the frequency is increased from zero. An examination of the equations given below for θ_o and ϕ_{ss} will indicate the limitations of this statement.

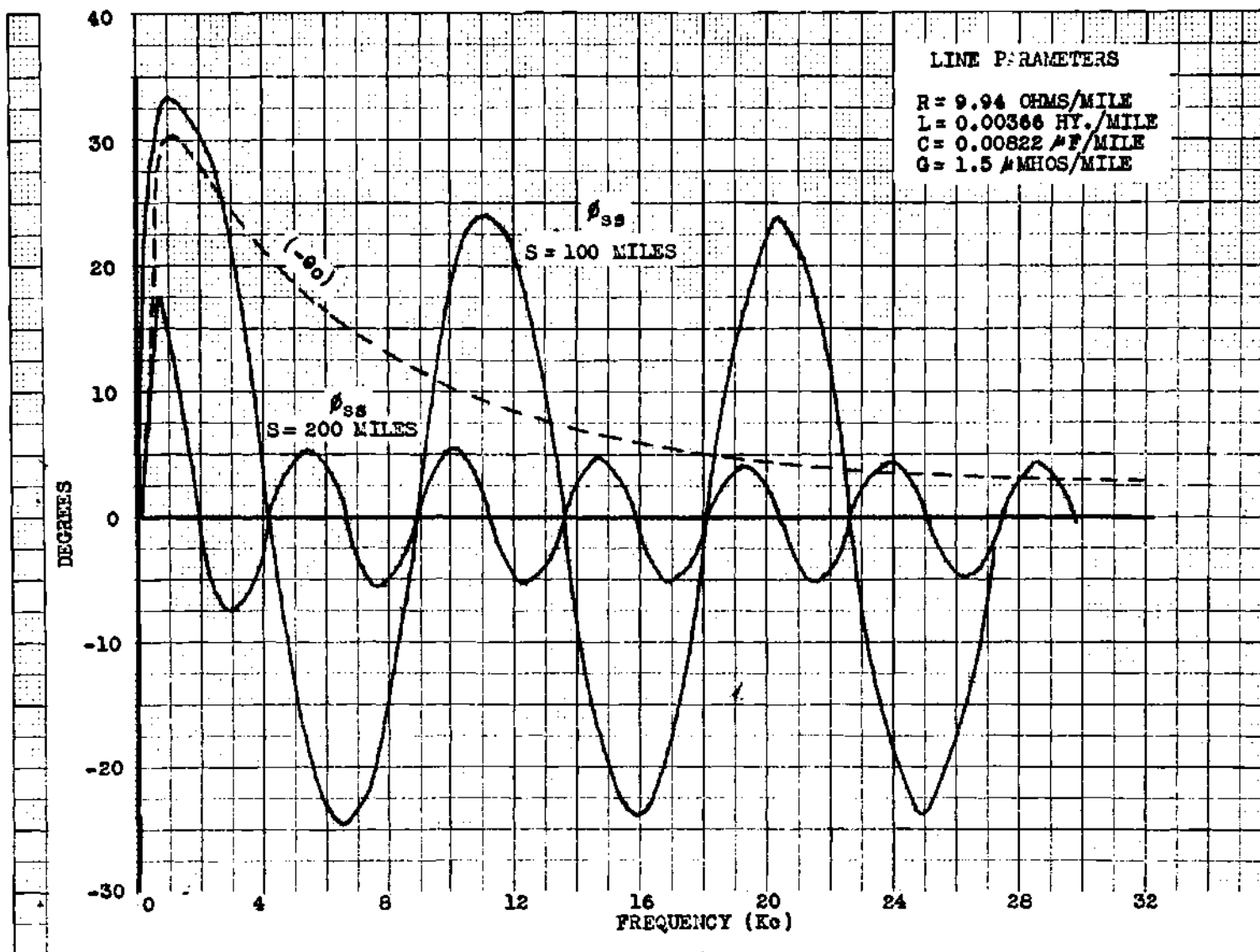


Fig. 10 $(-\theta_0)$ AND ϕ_{ss} VERSUS FREQUENCY

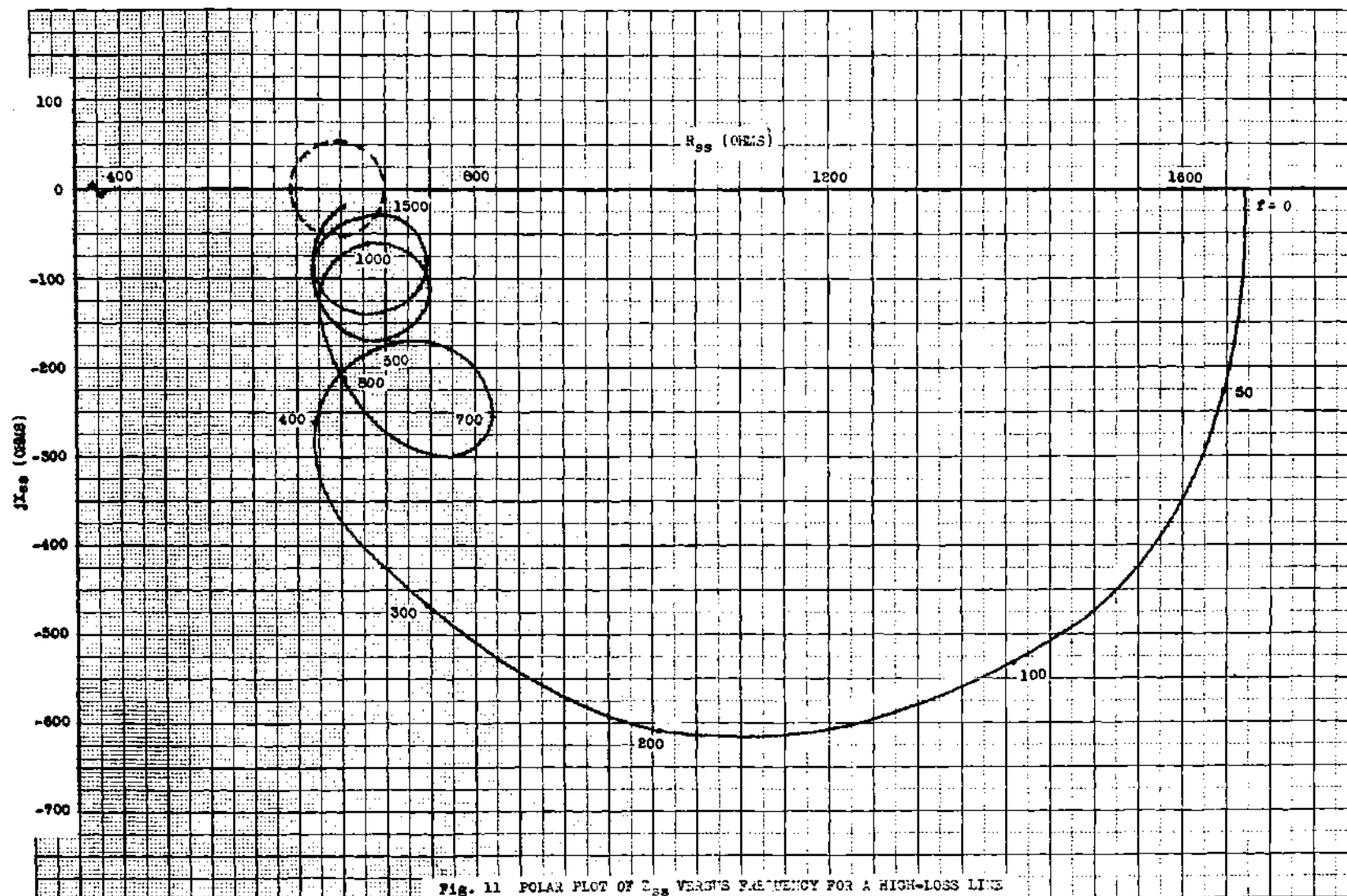


Fig. 11 POLAR PLOT OF Z_{22} VERSUS FREQUENCY FOR A HIGH-LOSS LINE

$$\theta_o = \frac{1}{2} \tan^{-1} \frac{\omega(LG - RC)}{RG + \omega^2 LC}$$

$$\phi_{ss} = \tan^{-1} \frac{\sin 2b}{\sinh 2a}$$

ϕ_{ss} is a positive angle at all frequencies below the frequency which makes $2b$ equal to π ; and θ_o , as stated before, is a negative angle at all frequencies for physical lines of interest. Since θ_{ss} is the sum of θ_o and ϕ_{ss} , it will be positive, and hence the input impedance of the line will be inductive, at the lowest frequencies only if $|\phi_{ss}|$ exceeds $|\theta_o|$ at these frequencies.

The equation for ϕ_{ss} shows that this angle will be large if the total attenuation of the line is small; and, conversely, that ϕ_{ss} will be small if the total attenuation is large. Figure 9, which shows the variation of $(-\theta_o)$ and ϕ_{ss} over the lower frequency range for a typical telephone line, indicates that $|\phi_{ss}|$ exceeds $|\theta_o|$ in this frequency range only for lengths of this particular line which are less than approximately 110 miles. For longer lengths of line $|\phi_{ss}|$ is less than $|\theta_o|$, θ_{ss} is negative, and the input impedance of the line is capacitive. Now lengths of this line greater than approximately 110 miles produce a total attenuation such that the condition (3-6) is not satisfied at all frequencies. Therefore, for lines whose total attenuation is such that the product of $\sinh 2a$ and $\tan |\theta_o|$ is not equal to or less than unity at all frequencies, the input impedance when the line is short-circuited is first capacitive as the frequency is increased from zero.

Condition (3-6) presents a convenient basis for classifying transmission lines as either low-loss or high-loss lines. Thus if a line has

parameters such that condition (3-6) is satisfied at all frequencies, it can be classified as a low-loss line. Likewise, if a line has parameters such that condition (3-6) is not satisfied at all frequencies, it can be classified as a high-loss line.

The Open-Circuited Line

The equation for θ_{so} , the angle of the input impedance, Z_{so} , of a line which is open-circuited is²⁰

$$\theta_{so} = \theta_o + \phi_{so} \quad (3-8)$$

where

$$\phi_{so} = \tan^{-1} \frac{-\sin 2b}{\sinh 2a} \quad (3-9)$$

Equation (3-9) shows that ϕ_{so} is negative until the frequency becomes high enough to make $\sin 2b$ negative. Since θ_{so} is the sum of two angles that are both negative at the frequencies just above zero, Z_{so} is first capacitive as the frequency is increased from zero. Thus, for high-loss lines the input impedance of the line is capacitive at the lowest frequencies whether the line is open-circuited or short-circuited.

The input impedance of an open-circuited line is purely resistive when

²⁰Weinbach, op. cit., p. 69.

$$\theta_{so} = \theta_o + \tan^{-1} \frac{-\sin 2b}{\sinh 2a} = 0. \quad (3-10)$$

Equation (3-10) can be arranged to yield

$$\tan |\theta_o| \sinh 2a \leq 1, \quad (3-11)$$

which gives the condition which must exist in order for θ_{so} to be zero.

This is the same condition required for θ_{ss} to be zero.

Figure 12 shows a plot of $(-\theta_o)$ and ϕ_{so} versus frequency. It is similar to Figure 10 for the short-circuited line. The difference between the two figures is brought about by the fact that ϕ_{so} is the negative of ϕ_{ss} . As a result of this, the frequencies at which θ_{so} is zero are different from those at which θ_{ss} is zero.

A polar plot of the input impedance of an open-circuited, high-loss line would be so similar to Figure 11 that it will not be shown. The general shape of the curve would be exactly the same; however, the magnitude and phase angle of Z_{so} would be different at all frequencies from the magnitude and phase angle of Z_{ss} .

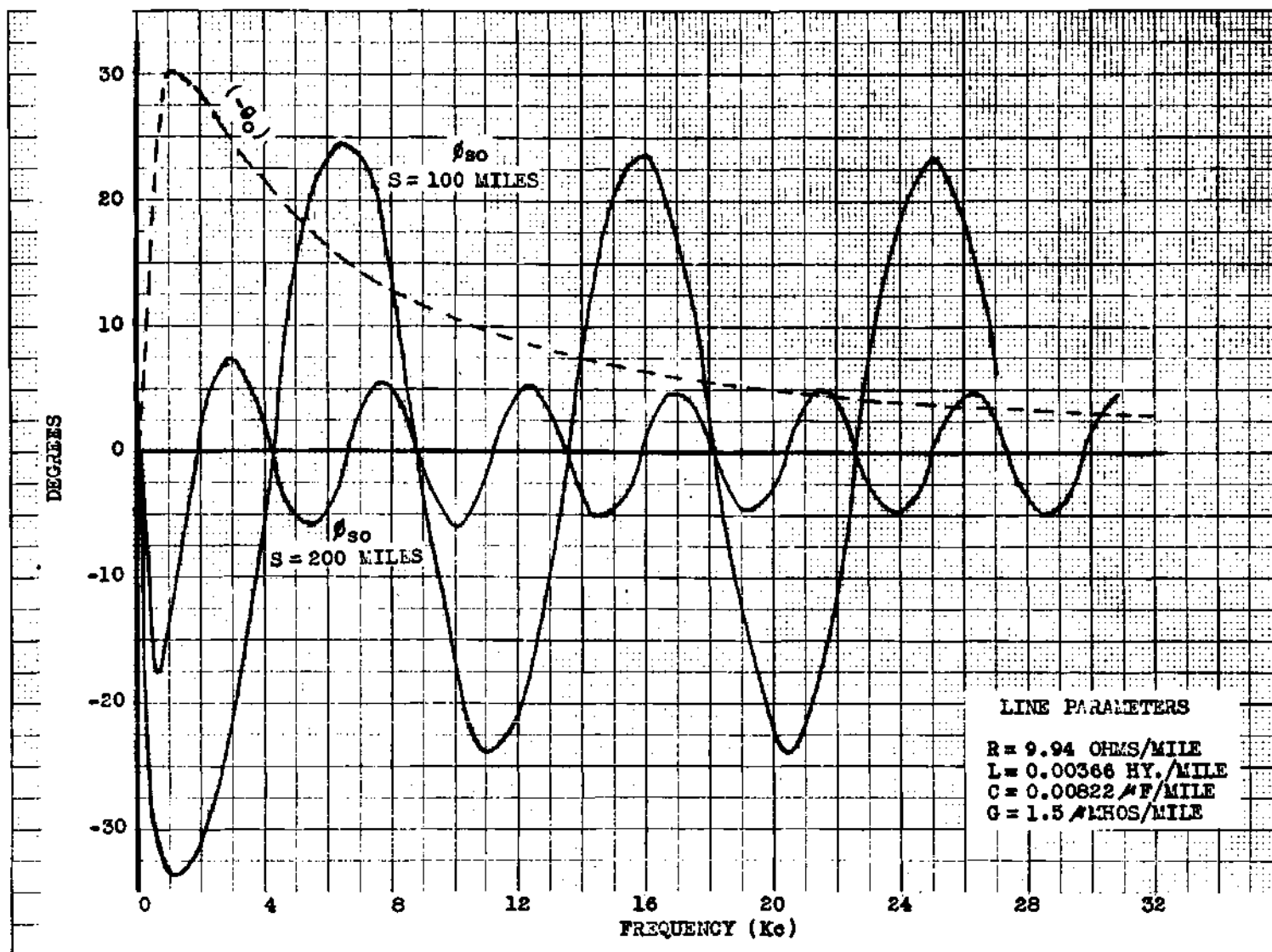


Fig. 12 $(-\theta_o)$ AND ϕ_{so} VERSUS FREQUENCY

CHAPTER V

THE DISTORTIONLESS LINE

The characteristics of the distortionless line are, in some instances, so different from those of the general finite- Q line that it merits special attention. Only the case of the short-circuited distortionless line will be considered, since the open-circuited case is quite similar to the short-circuited case.

The Propagation Constant²¹

The Attenuation per Mile. In general, the attenuation per mile increases with increasing frequency. Consider the equation for α :

$$\alpha = \sqrt{\frac{1}{2} \left[\sqrt{(RG + \omega^2 LC)^2 - (LG - RC)^2 \omega^2} + (RG - \omega^2 LC) \right]} \quad (4-1)$$

The distortionless line has values of R , L , G , and C such that

$$LG - RC = 0 \quad (4-2)$$

Under this condition, equation (4-1) reduces to

$$\alpha = \sqrt{RG} \quad (4-3)$$

This value of α is the minimum value for all lines. Equation (4-3) shows that α is a constant independent of frequency.

The Phase Shift per Mile. When equation (4-1) is substituted into the equation (1-6) for β , this equation reduces to

²¹Ware, op. cit., p. 83.

$$\beta = \omega \sqrt{\frac{L}{C}} \quad (4-4)$$

The Characteristic Impedance

Substitution of equation (4-1) into the equation for the characteristic impedance yields

$$Z_0 = \sqrt{\frac{L}{C}} \quad , \quad (4-5)$$

a purely resistive, constant value.

The Variation of Z_{ss} With Frequency

For the distortionless line, the frequencies at which Z_{ss} is a maximum or minimum are equally-spaced as is evident when $\beta = \omega \sqrt{LC}$ is substituted into equations (1-3) and (1-4). Equation (1-3) then becomes

$$(f_n)_{\max} = \frac{2n-1}{4S\sqrt{LC}} \quad (4-6)$$

Similarly, equation (1-4) becomes

$$(f_n)_{\min} = \frac{n}{2S\sqrt{LC}} \quad (4-7)$$

In this respect the distortionless line behaves like a lossless line.

The variation of the magnitudes of the maxima and minima of Z_{ss} will now be considered. Equation (1-2) yields for the magnitudes of the maxima of Z_{ss}

$$|Z_{ssn}|_{\max} = |Z_0|_n \coth \alpha_n S \quad , \quad (4-8)$$

where $|Z_{ssn}|_{\max}$, $|Z_0|_n$, and α_n are the magnitudes of Z_{ss} , Z_0 , and α

at the frequencies corresponding to the various values of n in equation (4-6). Since $|Z_o|$ and α are independent of frequency, the maxima of Z_{ss} will all be of the same magnitude. In this respect, too, the distortionless line behaves like a lossless line. The maxima of Z_{ss} are, however, of a lower magnitude for longer lengths (S) of the line.

For the magnitudes of the minima of Z_{ss} , equation (1-2) yields

$$|Z_{ssn}|_{\min} = |Z_o|_n \tanh \alpha_n S, \quad (4-9)$$

where $|Z_{ssn}|_{\min}$, $|Z_o|_n$, and α_n are the magnitudes of Z_{ss} , Z_o , and at the frequencies corresponding to the values of n in equation (4-7). The minima of Z_{ss} at the various frequencies are seen to be of constant magnitude because $|Z_o|$ and α are independent of frequency. Longer lengths of line, however, produce larger constant values of magnitude for the minima of Z_{ss} .

The Phase Angle of Z_{ss}

Since θ_o is zero at all frequencies for the distortionless line, the frequencies which produce maxima and minima of Z_{ss} are pure resistances.

The distortionless line is a low-loss line under the classification set forth in Chapter IV. This is true for all values of α and for all lengths of line. This phenomenon is possible because θ_o is zero at all frequencies. The right-hand side of equation (3-6) is, therefore, zero regardless of the value of α and regardless of the length of the line.

The input impedance of a short-circuited, distortionless line at

any frequency is given by equation (2-8) if a' is interpreted to be $S\sqrt{RG}$. The circles of Figure 3 are typical curves for the variation of Z_{ss} with frequency for different lengths of line. Longer lines produce the circles of smaller radius.

It is apparent that the closer the parameters of a finite-Q line approach the condition $LG - RC = 0$, the closer its characteristics approach those of a lossless line.

CHAPTER VI

SUMMARY AND CONCLUSIONS

This thesis presents an analysis of the characteristics of short-circuited and open-circuited transmission lines. The following conclusions are made:

1. The frequencies at which the input impedance is a maximum or a minimum are not equally-spaced. Furthermore, the maxima and minima are not purely resistive.
2. The frequencies at which the phase angle of the input impedance is zero are not equally-spaced.
3. At high frequencies where α approaches a constant value and the phase angle of the characteristic impedance approaches zero, a condition is approached where
 - (a) the frequencies at which the maxima and the minima of the input impedance occur are equally-spaced, and
 - (b) the frequencies at which the phase angle of the input impedance is zero are equally-spaced and coincide with the frequencies which produce a maximum value and a minimum value of input impedance.
4. A new criterion for classifying transmission lines as either low-loss or high-loss is based on the condition

$$\tan |\theta_0| \sinh 2\alpha \leq 1.$$

If a line has parameters (R, L, G, C, S) such that this condition is satisfied at all frequencies, the line is classified as a low-loss line; if a line

has parameters such that this condition is not satisfied at all frequencies, it is classified as a high-loss line.

5. Although the input impedance of a high-loss line passes through the various maximum and minimum values throughout the frequency spectrum, the expected points of unity power factor ($\theta_{ss} = 0$ or $\theta_{so} = 0$) do not occur at the low frequencies.

6. The input impedance of a high-loss line is capacitive at the lowest frequencies whether the line is open-circuited or short-circuited.

7. The distortionless line behaves like a lossless line in three respects:

- (a) the maxima of the input impedance are all of the same magnitude and the minima of the input impedance are all of the same magnitude;
- (b) the frequencies at which the input impedance is a maximum or a minimum are equally-spaced; and
- (c) the frequencies at which the phase angle of the input impedance is zero are equally-spaced and coincide with the frequencies at which the input impedance is a maximum and a minimum.

The analysis presented in this thesis is of particular importance to the communications field. It finds application in the use of high-loss lines for terminations on antennas²², in the use of transmission

²²Terman, F. E., *Radic Engineers' Handbook* (New York: McGraw-Hill Book Company, Inc., 1943), p. 804.

lines as dummy loads for antennas, and in fault location techniques.^{23,24}
It is applicable to communication lines in general and to long telephone lines in particular.

²³Honnell, M. A., "Location of Line Faults," Electronics, 17:110, 1944.

²⁴Blankmeyer, W. H., "Power Line Faults Location," Electronics, 17:166, 1944.

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